Indian Statistical Institute, Bangalore
B.Math (Hons.) I Year, Second Semester
Semestral Examination
Real Analysis II

Time: 3 hours | April 29, 2011 | Instructor: C.R.E.Raja
Maximum marks: 50

Section I: Answer all and each question is worth 2 Marks Total Marks 6

1. Let a and b be two points in a metric space X such that $a \neq b$. Prove that there is a $\delta > 0$ such that $N_\delta(a) \cap N_\delta(b) = \emptyset$.
2. Prove that compact subset of a metric space is bounded.
3. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be $g(x) = f(x) \cdot u$ for $x \in \mathbb{R}^n$ where $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and $u \in \mathbb{R}^m$. Then prove that $g'(x)(h) = f'(x)(h) \cdot u$ for $x, h \in \mathbb{R}^n$.

Section II: Answer any 4 and each question is worth 6 Marks Total Marks 24

1. Let E be a subset of a metric space X. Show that $x \in E'$ if and only if there is a sequence (x_n) in $E \setminus \{x\}$ such that $x_n \to x$.
2. If $f\colon [a, b] \to \mathbb{R}$ is a continuous function, then prove that $f \in \mathcal{R}[a, b]$.
3. Assume that $f \in \mathcal{R}[a, b]$, $f \geq 0$ and $\int_a^b f = 0$. Is $f = 0$? justify your answer. If further f is continuous on $[a, b]$, prove that $f = 0$.
4. Suppose $f \in \mathcal{R}[a, b]$ and ϕ is continuous on $\mathbb{R}$. Then show that $\phi \odot f \in \mathcal{R}[a, b]$.
5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives $D_1 f$ and $D_2 f$ on $\mathbb{R}^2$. If for each $x \in \mathbb{R}^2$, there is a $\eta_x > 0$ such that $D_1 f$ and $D_2 f$ are bounded in $N_{\eta_x}(x)$, then prove that f is continuous on $\mathbb{R}^2$.
6. (a) Let E be a convex open subset of $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^m$ be a differentiable function with bounded derivative. Then prove that there is a $M > 0$ such that $||f(x) - f(y)|| \leq M||x - y||$ for all $x, y \in E$.

   (b) If E is a connected open subset of $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^m$ is a differentiable function with $f'(x) = 0$ for all $x \in E$, then prove that f is constant on E.

Section III: Answer any 2 and each question is worth 10 Marks Total Marks 20

1. (a) Let (X, d) be a metric space and (a_n) and (b_n) be Cauchy sequences in X. Prove that $\{a_n \mid n \geq 1\}$ is bounded and $(d(a_n, b_n))$ converges in $\mathbb{R}$.

   (b) Let X be a metric space and $x \in X$. Prove that $x \in X'$ if and only if $\overline{X \setminus x} = X$.

2. (a) Let $f \in \mathcal{R}[a, b]$ and $\epsilon > 0$. Prove that there is a continuous function g on $[a, b]$ such that $\int_a^b |f - g| < \epsilon$.

   (b) Let $f \in \mathcal{R}[a, b]$ and F be a differentiable function on $[a, b]$ such that $F' = f$. Then prove that $\int_a^b f = F(b) - F(a)$.

3. Let E be an open subset of $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^m$ be a differentiable function. For $1 \leq i \leq m$, $f_i\colon E \to \mathbb{R}$ is such that $f(x) = (f_1(x), \cdots, f_m(x))$ for all $x \in E$.

   (a) Prove that $D_j f_i$ exists at x and $f'(x)(e_j) = \sum_{i=1}^m D_j f_i(x) u_i$ for $x \in E$.

   (b) Let $E = \mathbb{R}$ and f have second order partial derivatives on E. If $||f(t)|| = |t|$ and $||f'(t)|| = 1$ for all $t \in \mathbb{R}$, prove that for $t \in \mathbb{R}$, $f^{(2)}(t) \cdot f(t) = 0$ where $f^{(2)}(t) = (D_{11} f_1(t), \cdots, D_{11} f_m(t))$.